B.Sc-I, Paper-I, Group B. 1

Mathematical Treatment of the phenomenon of oscillatory Motion:—

We have treated the phenomenon of oscillatory motion only experimentally. We shall now investigate the dynamical basis for such motion.

The motion of a simple pendulum or that of a vibrating string of a violin. The system when displaced from its stationary position would return to it on account of the restoring force called into play but the momentum acquired carries it past that position to the other side till its kinetic energy is wholly converted to potential and the system comes to momentary rest. Again the force brings it back and the to-and fro motion is repeated which, but for resistance may continue indefinitely. In all such cases an interchange of kinetic energy to potential energy and vice versa, is taking place. Assuming that neither energy is received nor dissipated into heat by internal friction etc. We have from the law of conservation of energy that the sum of kinetic energy and potential energy will be constant.

Let us consider the oscillatory motion of such a conservation system, say of a mass m under the influence of a restoring force which is $-\mu$ times the displacement. Then the K.E & potential energies are

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 \quad — (1)$$

and $$P = \int_0^y \mu y\,dy = \frac{1}{2}\mu y^2 \quad — (2)$$

Since there is no loss or supply of energy by some external agency we have, $K+P = \text{Constant}$

$$\text{or, } \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}\mu y^2 = C \quad — (3)$$

Differentiating w.r.to t, we have,

$$m\frac{dy}{dt}\cdot\frac{d^2y}{dt^2}+\mu y\frac{dy}{dt}=0$$

or, $m\frac{d^2y}{dt^2}+\mu y=0$

or, $\frac{d^2y}{dt^2}+n^2y=0$ —— (5)

where, $n^2=\frac{\mu}{m}$

This is the Differential equation of the motion sought. The integration of this equation will give us the displacement of the particle under the influence of the restoring force. Multiplying this by $2\frac{dy}{dt}$ we have,

$$2\frac{dy}{dt}\frac{d^2y}{dt^2}+2n^2y\frac{dy}{dt}=0$$

integrating we get, suppose $y=a$, when $\frac{dy}{dt}=0$.

then, $c'=a^2n^2$

$\therefore\left(\frac{dy}{dt}\right)^2=a^2n^2-n^2y^2=n^2(a^2-y^2)$

$\frac{dy}{dt}=n\sqrt{a^2-y^2}$

or, $\frac{dy}{\sqrt{a^2-y^2}}=n\,dt$

integrating again,

$\sin^{-1}\frac{y}{a}=nt+\phi$

$\frac{y}{a}=\sin(nt+\phi)$

$\boxed{y=a\sin nt+\phi}$

Mathematical treatment:

Consider a plane longitudinal wave travelling in a long tube of unit area of cross-section.

The simple harmonic wave be represented by

$$y = a \sin \frac{2\pi}{\lambda}(vt - x) \quad \text{--- (1)}$$

Diff. w.r. to t, obtain the velocity of all the particles in the oscillating layer (x = constant).

$$V = \frac{dy}{dt} = \frac{2\pi v}{\lambda} a \cos \frac{2\pi}{\lambda}(vt - x) \quad \text{--- (2)}$$

Now, the mass of unit area of a layer of thickness δx is $\rho \delta x$ where ρ is the undisturbed density of the medium.

The K.E of the layer is given by

$$\delta K = \frac{1}{2} \text{mass} \times (\text{velocity})^2$$

$$= \frac{1}{2} \rho \delta x \left(\frac{dy}{dt}\right)^2$$

$$= \frac{1}{2} \rho \delta x \left(\frac{2\pi a v}{\lambda}\right)^2 \cos^2 \frac{2\pi}{\lambda}(vt - x) \quad \text{--- (3)}$$

The average kinetic energy of the whole wave of length l and unit cross-section is

$$K = \frac{1}{2} \rho \left(\frac{2\pi a v}{\lambda}\right)^2 \int_0^l \cos^2 \frac{2\pi}{\lambda}(vt - x)\, dx$$

$$= \frac{1}{4} \rho \left(\frac{2\pi a v}{\lambda}\right)^2 \int_0^l \left[1 + \cos \frac{4\pi}{\lambda}(vt - x)\right] dx \quad \text{--- (4)}$$

The mean value of the 'cos' term under the integral sign is zero. Hence the average K.E per unit area per unit length (i.e per unit volume) of the wave is

$$\boxed{K = \frac{1}{4} \rho \left(\frac{2\pi a v}{\lambda}\right)^2} \quad \text{--- (5)}$$

Now, obtain an expression for the potential energy wave.

The potential energy of a layer of unit cross-section and thickness δx is

δP = workdone during compression per unit volume × volume of the layer.

$= \frac{1}{2}$ stress × strain × volume of the layer.

$$= \frac{1}{2} E \times \frac{dy}{dx} \cdot \frac{dy}{dx} \times \delta x \qquad \left\{ \because \text{Stress} = \text{elasticity} \times \text{strain} = E \times \frac{dy}{dx} \right\}$$

$$= \frac{1}{2} E \delta x \left(\frac{dy}{dx} \right)^2$$

$$= \frac{1}{2} \rho v^2 \delta x \left(\frac{dy}{dx} \right)^2 \qquad \left[\because v = \sqrt{\frac{E}{\rho}} \therefore E = \rho v^2 \right]$$

But $\frac{dy}{dx} = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x)$

$$\therefore \delta P = \frac{1}{2} \rho v^2 \delta x \left[\left\{ -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x) \right\}^2 \right.$$

$$\boxed{\delta P = \frac{1}{2} \rho \delta x \left(\frac{2\pi a v}{\lambda} \right)^2 \cos^2 \frac{2\pi}{\lambda}(vt - x)} \quad \text{— (6)}$$

which is the same as the kinetic energy δK.

It is maximum at the points at which $\frac{dy}{dt}$ the layer velocity is maximum i.e at the centre of a compression or the centre of a rarefaction.

Proceeding in the same manner as in the case of kinetic energy we find that the average potential energy per unit volume of the ~~wh~~ whole wave is also given by

③

$$P = \frac{1}{4} \rho \left(\frac{2\pi a v}{\lambda}\right)^2 \quad (9)$$

i.e the energy of a progressive wave at any instant is half kinetic and half potential.

⊗ Considering the average values of the kinetic and potential energies, we find that the average total energy of the wave per unit volume is

$$E = K + P$$

$$= \frac{1}{2} \rho \left(\frac{2\pi a v}{\lambda}\right)^2 \quad \{\text{where } v = n\lambda$$

$$\therefore n = \frac{v}{\lambda}$$

$$= \frac{1}{2} \rho (2\pi a n)^2$$

$$= 2\rho\pi^2 a^2 n^2$$

Hence the energy transferred per second must be equal to energy contained in length v. This is the rate of flow of energy per unit cross-section passing along the line of advance of the waves.

This may be regarded as energy current or the intensity of the sound wave. It is given by

$$C = E \cdot v$$

$$= \frac{1}{2} \rho (2\pi a n)^2 \times v$$

$$\boxed{C = 2\rho v \pi^2 a^2 n^2}$$

it is directly proportional to the density of the medium and the wave velocity and to the square of the amplitude and the frequency of vibration. it may also be noted that though the K.E and P.E both vary x and t, yet the total energy E and the energy current C are both independent

Wave Intensity :- The intensity I of a wave is defined as the amount of energy incident per second unit area perpendicular to the direction of wave propagation. Hence it is equal to the energy current.

$$\therefore \quad \boxed{I = 2\rho v \pi^2 a^2 n^2}$$

Thus the intensity is equal to the product of energy density and wave velocity. It is direclty proportional to the density of the density of the medium and the wave velocity, and to the square of the amplitude and the square of the frequeny.